WARWICK
THE UNIVERSITY OF WARWICK

Original citation:
Sun, Tao, Yang, Shuofei , Huang , Tian and Jian S., Dai. (2016) A way of relating instantaneous and finite screws based on the screw triangle product. Mechanism and Machine Theory, 108. pp. 75-82.

Permanent WRAP URL:
http://wrap.warwick.ac.uk/82164

Copyright and reuse:
The Warwick Research Archive Portal (WRAP) makes this work by researchers of the University of Warwick available open access under the following conditions. Copyright © and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable the material made available in WRAP has been checked for eligibility before being made available.

Copies of full items can be used for personal research or study, educational, or not-for-profit purposes without prior permission or charge. Provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

Publisher's statement:
© 2016, Elsevier. Licensed under the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International http://creativecommons.org/licenses/by-nc-nd/4.0/

A note on versions:
The version presented here may differ from the published version or, version of record, if you wish to cite this item you are advised to consult the publisher's version. Please see the 'permanent WRAP URL' above for details on accessing the published version and note that access may require a subscription.

For more information, please contact the WRAP Team at: wrap@warwick.ac.uk

warwick.ac.uk/lib-publications

A Way of Relating Instantaneous and Finite Screws based on the Screw Triangle Product

Tao Sun [1], Shuofei Yang[1], Tian Huang[1,2,*], Jian S. Dai[3]

[1] Key Laboratory of Mechanism Theory and Equipment Design of Ministry of Education, Tianjin University, Tianjin 300350, China

[2] School of Engineering, The University of Warwick, Coventry CV4 7AL, UK

[3] Centre for Robotics Research, School of Natural Sciences and Mathematics, King's College London, University of London, London WC2R 2LS, UK

Abstract: It has been a desire to unify the models for structural and parametric analyses and design in the field of robotic mechanisms. This requires a mathematical tool that enables analytical description, formulation and operation possible for both finite and instantaneous motions. This paper presents a method to investigate the algebraic structures of finite screws represented in a quasi-vector form and instantaneous screws represented in a vector form. By revisiting algebraic operations of screw compositions, this paper examines associativity and derivative properties of the screw triangle product of finite screws and produces a vigorous proof that a derivative of a screw triangle product can be expressed as a linear combination of instantaneous screws. It is proved that the entire set of finite screws forms an algebraic structure as Lie group under the screw triangle product and its time derivative at the initial pose forms the corresponding Lie algebra under the screw cross product, allowing the algebraic structures of finite screws in quasi-vector form and instantaneous screws in vector form to be revealed.

Keywords: Robotic mechanisms, Screw theory, Finite screw, Screw triangle, Lie group and Lie algebra

1. Introduction

The instantaneous screw [1] has been considered as a powerful tool and extensively applied in analysis and design of serial and parallel mechanisms. By taking twist of an end link as a linear combination of instantaneous screws produced by joints, Hunt [2] and Duffy [3] developed a method to formulate Jacobian matrix. Angeles [4] presented an alternative way to do so by taking all actuation wrenches as instantaneous screws. These works were extended by Joshi and Tsai [5], Huang and Liu [6] in dealing with lower mobility serial and parallel mechanisms, resulting in the overall Jacobian and generalized Jacobian matrix that allow velocity, force, stiffness and rigid body dynamic analyses to be unified using instantaneous screw theory. Another use of screw theory in analysis and synthesis of robotic mechanisms is mobility analysis and type synthesis. Regarding small and finite displacements or velocities and constraint forces of a mechanism in a systematic frame of a screw system and its reciprocal system, Dai and Rees Jones [7] revealed the interrelationship and intersection of all screw systems and their reciprocal systems by introducing set theory in screw theory. In the meantime, they [8] developed a linear algebraic procedure to obtain a reciprocal screw system and its basis in a null space construction using cofactors from a screw algebra context. These theories were directly used in mobility analysis of different types of parallel mechanisms, including those having overconstraints [9]. Utilizing annihilator property between a screw system and its reciprocal system, Huang and Li [10], Fang [11], Kong [12] and colleagues proposed a number of simple and effective approaches for type synthesis of various parallel mechanisms. However, since instantaneous screws are not valid to represent finite motions, final verification is required to ensure consistency between finite and instantaneous motions [13-15].

Corresponding Author, Email: tianhuang@tju.edu.cn

In order to describe finite motions precisely, the finite screw was proposed. Investigation into finite screws can be traced back to the pioneering work of Dimentberg [16-17] and Roth [18] in dealing with composition of finite motions of a rigid body, leading to development of the screw triangle product that provides a sound geometric interpretation to the Rodrigues formula with dual angles [17]. Investigations into finite screws were developed by Roth [18], Parkin [19], Huang [20-21] and Dai [22-23]. They proposed a simple and widely accepted form of finite screws that can be analogously expressed as a quasi-vector with six algebraic entries. A finite screw in this form looks extraordinarily similar to an instantaneous one even if it is nonlinear in nature. This form allows the linear format of a screw triangle to be made in such a way that the composition of two finite screws can be expressed as the sum of five meaningful terms [21, 24]. Based upon the algebraic characteristic of finite screws, Dai, Holland and Kerr [22] and Dai [25-26] revealed the relationship between finite screws and instantaneous screws through solving the eigenscrew and derivative of the finite displacement screw matrix [25]. Utilizing finite screw theory, finite motion analyses of different geometrical elements, such as points, lines and planes, as well as simple open loop and closed loop mechanisms are carried out by Huang [20, 27], Hunt [28].

It has been a desire to develop a theoretical package that enables analysis and synthesis of robotic mechanisms to be integrated into a unified and consistent process [29]. This issue needs to relate nonlinear finite to linearized instantaneous motions of rigid body systems. Hence, a preliminary and essential step to achieve the aforementioned goal is to develop a general and effective method that enables the description, formulation and operation of finite and instantaneous motions to be implemented under a consistent and unified mathematical tool, a fundamental and challenging issue in the field of mechanisms and robotics. The methods available at hand can be roughly divided into three categories, i.e. matrix group based method, dual quaternion based method and screw theory based method.

The matrix group based method can be traced back to the Erlangen program proposed by Klein [30]. By utilizing matrix groups to describe finite motions of rigid body systems, Brockett [31] applied the exponential map between Lie group SE(3) and Lie algebra se(3) to relating models for finite motions to that for instantaneous motions [32]. However, two barriers are encountered in the use of matrix groups for finite motion composition. The first barrier arises from implementation of matrix groups for affine transformations where finite motions of a rigid body cannot be directly represented by Chasles' axis [17] as well as by the angular and/or linear displacement about the axis, leading to a complicated description of rigid body motion. The second barrier comes from that the finite motion composition cannot be algebraically derived by Baker-Campbell-Hausdorff formula [33]. Consequently, motion patterns of a number of parallel mechanisms cannot precisely be described using the existing matrix group based method since they can no longer be represented by products of several Lie subgroups [34].

The dual quaternion based method can be traced back to description of rotations of a rigid body by means of Euler's four-square identity, Euler-Rodrigues parameters [23, 35] and Hamilton quaternions [36]. Perez and McCarthy [37] seem to be the first to use the dual quaternions to do analyses for finite and instantaneous motions of serial kinematic chains. In their work, unit dual quaternions and unit pure dual quaternions were used for describing finite and instantaneous motions, for the algebraic structure of the former is a double cover of Lie group SE(3) whose Lie algebra in turn constitutes the latter as by Selig [38]. With the aid of group theory, Selig [39-40] and Dai [26, 35] investigated the algebraic properties of the exponential and Cayley maps between unit dual quaternions and unit pure dual quaternions. By introducing the notation of high-dimensional Clifford algebra, Selig [40] and Featherstone [41] extended the dual quaternions representation to deal with rigid body dynamics. However, a unit dual quaternion is not the simplest form of a rigid body motion. The redundancy in dual quaternion representation may cause complexity in analytical expressions of the finite motion operations. In addition, the Rodrigues formula with dual angles is not the simplest form of the Baker-Campbell-Hausdorff formula when it is applied to composition of finite motions of a rigid body [42].

The screw theory based method depends on the development of instantaneous screws and finite screws. Considering the algebraic characteristic of finite screws, Dai [22] demonstrated the relationship between finite displacement screw operation and the different matrix representations of SE(3) elements as well as quaternions [25, 35]. By solving the eigenscrew and derivative of finite displacement screw matrix, Dai [25] formulated the eigen and differential mappings between finite screws and instantaneous screws. Meanwhile, the consistency between these mappings and the exponential mapping of matrix Lie group/Lie algebra or the Euler-Rodrigues formula was revealed [25-26, 35]. The correspondence between vector subspaces and screw systems was discussed by Huang, Sugimoto and Parkin [43] to differentiate finite screw systems from screw systems arising from instantaneous kinematics and statics. Although the composition of finite motions can be expressed as a screw triangle product of finite screws in quasi-vector form [21, 24], and that of instantaneous motions as the linear combination of instantaneous screws in vector form, the algebraic structures of two kinds of screws remain an open issue to be investigated.

Addressing on the need to integrate models for analysis and synthesis of robotic mechanisms in a unified mathematic framework, this paper intends to reveal algebraic structures of finite screws in a quasi-vector form and instantaneous screws in a vector form. The paper is organized as follows. Having a brief review of the state-of-the-art of finite and instantaneous screw theory in Section 1, Section 2 presents the derivation of finite screws from dual quaternions and addresses the description of finite screws in quasi-vector form and instantaneous screws in vector form. Section 3 explores the associativity and derivative properties of the screw triangle product, resulting in that the set of finite screws is an associative and differentiable algebraic structure. This leads to a vigorous proof in Section 4 that the entire set of finite screws forms a Lie group under the screw triangle product and its time derivative at the identity forms the corresponding Lie algebra under the screw cross product before the conclusions are drawn in Section 5.

2. The Description of Finite and Instantaneous Screws

2.1 The derivation of finite screws from dual quaternions

The finite motion of a body from its initial pose to an arbitrary pose can be parameterized as a unit dual quaternion [26, 36]

$$\boldsymbol{D} = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\boldsymbol{S} \tag{1}$$

where $\boldsymbol{S} = \boldsymbol{s}_f + \varepsilon\left(\boldsymbol{r}_f \times \boldsymbol{s}_f\right)$ is the dual vector describing the finite motion axis (the Chasles' axis), $\boldsymbol{s}_f$ is the unit vector of this axis, $\boldsymbol{r}_f$ is the position vector pointing from the fixed reference point to an arbitrary point on the axis, ε is defined as the dual unit with the property $\varepsilon^2 = 0$; $\hat{\theta} = \theta + \varepsilon t$ is the dual angle, θ and t are the angular displacement about and linear displacement along the Chasles' axis with respect to the initial pose.

The linearization of sine and cosine of $\hat{\theta}$ allows $\boldsymbol{D}$ to be expressed as the sum of a dual scalar and a dual vector

$$\boldsymbol{D} = \left(\cos\frac{\theta}{2} - \varepsilon\frac{t}{2}\sin\frac{\theta}{2}\right) + \left(\sin\frac{\theta}{2}\boldsymbol{s}_f + \varepsilon\left(\sin\frac{\theta}{2}\left(\boldsymbol{r}_f \times \boldsymbol{s}_f\right) + \frac{t}{2}\cos\frac{\theta}{2}\boldsymbol{s}_f\right)\right) \tag{2}$$

It is easy to see that the dual vector contains all the elements, i.e. the Chasles' axis, the angular and linear displacement necessary to describe a finite motion. Thus, dividing the dual vector by $0.5\cos\left(\theta/2\right)$ leads to a finite displacement screw (or finite screw for simplicity) in six-dimensional quasi-vector form, which can be considered as the non-redundant minimal description of finite motion.

$$\boldsymbol{S}_f = 2\tan\frac{\theta}{2}\begin{pmatrix}\boldsymbol{s}_f \\ \boldsymbol{r}_f \times \boldsymbol{s}_f\end{pmatrix} + t\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_f\end{pmatrix} \tag{3}$$

2.2 The relationship between a finite screw and an instantaneous screw

In this section, we develop the relationship between a single finite screw in quasi-vector form and a single instantaneous screw in vector form. Assuming that the arbitrary pose of a body varies with time, the time derivative of $\boldsymbol{S}_f$ gives

$$\dot{\boldsymbol{S}}_f = \frac{\dot{\theta}}{\cos^2\left(\frac{\theta}{2}\right)}\begin{pmatrix}\boldsymbol{s}_f \\ \boldsymbol{r}_f \times \boldsymbol{s}_f\end{pmatrix} + 2\tan\frac{\theta}{2}\begin{pmatrix}\dot{\boldsymbol{s}}_f \\ \dot{\boldsymbol{r}}_f \times \boldsymbol{s}_f + \boldsymbol{r}_f \times \dot{\boldsymbol{s}}_f\end{pmatrix} + \dot{t}\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_f\end{pmatrix} + t\begin{pmatrix}\boldsymbol{0} \\ \dot{\boldsymbol{s}}_f\end{pmatrix} \tag{4}$$

where $\dot{\theta}$ and $\dot{t}$ are the angular velocity about and linear velocity along the screw axis at the instant considered. Note that at the initial pose where $\theta = 0$ and $t = 0$, the finite displacement axis (the Chasles's axis) is coincident with the instantaneous velocity axis (the instantaneous screw axis or the Mozzi's axis) at the instant. Thus, Eq. (4) can be rewritten as

$$\dot{\boldsymbol{S}}_f\Big|_{\substack{\theta=0 \\ t=0}} = \dot{\theta}\begin{pmatrix}\boldsymbol{s}_t \\ \boldsymbol{r}_t \times \boldsymbol{s}_t\end{pmatrix} + \dot{t}\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_t\end{pmatrix} \tag{5}$$

This means that the time derivative of $\boldsymbol{S}_f$ at the initial pose is exactly the instantaneous screw (or twist) $\boldsymbol{S}_t$ at the same pose

$$\boldsymbol{S}_t = \dot{\boldsymbol{S}}_f\Big|_{\substack{\theta=0 \\ t=0}} = \begin{pmatrix}\boldsymbol{\omega} \\ \boldsymbol{v}\end{pmatrix} \tag{6}$$

where $\boldsymbol{s}_t$ denotes the unit vector of the instantaneous screw axis, $\boldsymbol{r}_t$ denotes the position vector pointing from the fixed reference point to an arbitrary point on the axis. $\boldsymbol{\omega}$ and $\boldsymbol{v}$ should be understood as the angular velocity of the body and the linear velocity of the point instantaneously coincident with the fixed reference point at the initial pose, and $\boldsymbol{s}_t$, $\boldsymbol{r}_t$, $\dot{\theta}$ and $\dot{t}$ appearing in Eq. (5) should be understood as the corresponding quantities at the same pose.

3. Associativity and Derivative Laws of Screw Triangle Products

Lie group theory [44-45] shows that a representation of a group should be a subgroup of a general linear group acting on a vector space. In this sense, the entire set of finite screws in 6×1 quasi-vector form is neither SE(3) nor its representation because a finite screw is not a linear transformation acting on any point or line coordinate systems. Therefore, unlike matrix or dual quaternion representations of SE(3) or its double cover, the set of finite screws cannot simply inherit all the properties of SE(3). Thus, the algebraic structure of this set under screw triangle product needs to be revealed. For a given set and a product, four check points must be examined to show that this is a group, i.e. closure, associativity, existence of an identity, and existence of inverses. In addition, differentiability should be ensured to prove it is a Lie group. In this section, we firstly investigate closure and associativity of finite screws under the screw triangle product. Then, we will investigate the relationship between finite motions generated by a number of finite screws and instantaneous motions produced by the same number of instantaneous screws. The outcomes are useful for revealing algebraic structures of finite screws in quasi-vector form and instantaneous screws in vector form.

Assume that the finite motion of a rigid body is generated by a number of successive one-parameter finite screws (1-DOF finite motion). Each has the form given in Eq. (3)

$$\boldsymbol{S}_{f,j} = 2\tan\frac{\theta_j}{2}\begin{pmatrix}\boldsymbol{s}_{f,j}\\ \boldsymbol{r}_{f,j}\times\boldsymbol{s}_{f,j}\end{pmatrix} + t_j\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_{f,j}\end{pmatrix},\quad j = 1,2,3,\cdots \tag{7}$$

where $\theta_j \equiv 0$ for pure translation, otherwise the ratio $t_j/\theta_j = h_j$ is set to be a constant for screw motion. Note that $\boldsymbol{s}_{f,j}$ and $\boldsymbol{r}_{f,j}$ are constant vectors since the finite motion is accounted from the initial pose where $\boldsymbol{s}_{f,j} = \boldsymbol{s}_{t,j}$ and $\boldsymbol{r}_{f,j} = \boldsymbol{r}_{t,j}$.

The composition of two successive finite screws can be derived from the algebraic product of two dual quaternions $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ which have the form given in Eq. (1) [26, 36]

$$\boldsymbol{D}_{12} = \boldsymbol{D}_2\boldsymbol{D}_1 = \cos\frac{\hat{\theta}_{12}}{2} + \sin\frac{\hat{\theta}_{12}}{2}\boldsymbol{S}_{12} \tag{8}$$

where

$$\cos\frac{\hat{\theta}_{12}}{2} = \cos\frac{\hat{\theta}_1}{2}\cos\frac{\hat{\theta}_2}{2} - \sin\frac{\hat{\theta}_1}{2}\sin\frac{\hat{\theta}_2}{2}\boldsymbol{S}_1\square\boldsymbol{S}_2 \tag{9}$$

$$\sin\frac{\hat{\theta}_{12}}{2}\boldsymbol{S}_{12} = \sin\frac{\hat{\theta}_1}{2}\cos\frac{\hat{\theta}_2}{2}\boldsymbol{S}_1 + \cos\frac{\hat{\theta}_1}{2}\sin\frac{\hat{\theta}_2}{2}\boldsymbol{S}_2 + \sin\frac{\hat{\theta}_1}{2}\sin\frac{\hat{\theta}_2}{2}\boldsymbol{S}_2\times\boldsymbol{S}_1 \tag{10}$$

Linearizing sine and cosine of the dual angles, and equating the real and dual parts on both sides of Eq. (10), gives

$$\sin\frac{\theta_{12}}{2}\boldsymbol{s}_{12} = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\boldsymbol{s}_{f,1} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\boldsymbol{s}_{f,2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(\boldsymbol{s}_{f,2}\times\boldsymbol{s}_{f,1}\right) \tag{11}$$

$$\begin{aligned}
&\sin\frac{\theta_{12}}{2}\left(\boldsymbol{r}_{f,12}\times\boldsymbol{s}_{f,12}\right) + \frac{t_{12}}{2}\cos\frac{\theta_{12}}{2}\boldsymbol{s}_{f,12}\\
&= \left(\frac{t_1}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \frac{t_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\boldsymbol{s}_{f,1} + \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(\boldsymbol{r}_{f,1}\times\boldsymbol{s}_{f,1}\right) + \left(\frac{t_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \frac{t_1}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\boldsymbol{s}_{f,2}\\
&+ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(\boldsymbol{r}_{f,2}\times\boldsymbol{s}_{f,2}\right) + \left(\frac{t_2}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \frac{t_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\left(\boldsymbol{s}_{f,2}\times\boldsymbol{s}_{f,1}\right)\\
&+ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(\boldsymbol{s}_{f,2}\times\left(\boldsymbol{r}_{f,1}\times\boldsymbol{s}_{f,1}\right) + \left(\boldsymbol{r}_{f,2}\times\boldsymbol{s}_{f,2}\right)\times\boldsymbol{s}_{f,1}\right)
\end{aligned} \tag{12}$$

Dividing both sides of Eqs. (11)-(12) by $0.5\cos\left(\theta_{12}/2\right)$, respectively, and rewriting them into a finite screw form, finally results in

$$\boldsymbol{S}_{f,12} = \boldsymbol{S}_{f,1}\square\boldsymbol{S}_{f,2} = \frac{1}{1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2}}\left(\boldsymbol{S}_{f,1} + \boldsymbol{S}_{f,2} - \frac{1}{2}\boldsymbol{S}_{f,1}\times\boldsymbol{S}_{f,2} - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\left(t_2\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_{f,1}\end{pmatrix} + t_1\begin{pmatrix}\mathbf{0}\\ \boldsymbol{s}_{f,2}\end{pmatrix}\right)\right) \tag{13}$$

where symbol "$\square$" is referred to as screw triangle product [21], and $\boldsymbol{S}_{f,1}\times\boldsymbol{S}_{f,2}$ is defined as the screw cross product of $\boldsymbol{S}_{f,1}$ and $\boldsymbol{S}_{f,2}$

$$\boldsymbol{S}_{f,1}\times\boldsymbol{S}_{f,2}=\begin{pmatrix} 2\tan\frac{\theta_1}{2}\boldsymbol{s}_{f,1}\times 2\tan\frac{\theta_2}{2}\boldsymbol{s}_{f,2} \\ 2\tan\frac{\theta_1}{2}\boldsymbol{s}_{f,1}\times\left(2\tan\frac{\theta_2}{2}\boldsymbol{r}_{f,2}\times\boldsymbol{s}_{f,2}+t_2\boldsymbol{s}_{f,2}\right)+\left(2\tan\frac{\theta_1}{2}\boldsymbol{r}_{f,1}\times\boldsymbol{s}_{f,1}+t_1\boldsymbol{s}_{f,1}\right)\times 2\tan\frac{\theta_2}{2}\boldsymbol{s}_{f,2}\end{pmatrix} \tag{14}$$

Obviously, $\boldsymbol{S}_{f,12}$ satisfies the closure of finite motion composition since it can be rewritten into the standard form given by Eq. (3) *via* adequate algebraic operation.

3.1 The associativity law of successive screw triangle products

The associativity law of composition of finite screws can directly be proved using Eq. (13). Consider the composition of three arbitrary finite screws

$$\boldsymbol{S}_{f,123}=\boldsymbol{S}_{f,1}\triangle\boldsymbol{S}_{f,2}\triangle\boldsymbol{S}_{f,3} \tag{15}$$

By assuming that $\boldsymbol{S}_{f,12}=\boldsymbol{S}_{f,1}\triangle\boldsymbol{S}_{f,2}$ is the screw triangle product of the first two, the composition of $\boldsymbol{S}_{f,12}$ and $\boldsymbol{S}_{f,3}$ yields

$$\boldsymbol{S}_{f,(12)3}=\boldsymbol{S}_{f,12}\triangle\boldsymbol{S}_{f,3} \tag{16}$$

Hence, visualizing $\boldsymbol{S}_{f,12}$ as the first finite screw given in Eq. (13), leads to

$$\boldsymbol{S}_{f,(12)3}=\frac{1}{1-\tan\frac{\theta_{12}}{2}\tan\frac{\theta_3}{2}\boldsymbol{s}_{f,12}^{\mathrm{T}}\boldsymbol{s}_{f,3}}\left(\boldsymbol{S}_{f,12}+\boldsymbol{S}_{f,3}-\frac{1}{2}\boldsymbol{S}_{f,12}\times\boldsymbol{S}_{f,3}-\tan\frac{\theta_{12}}{2}\tan\frac{\theta_3}{2}\left(t_3\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,12}\end{pmatrix}+t_{12}\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{f,3}\end{pmatrix}\right)\right) \tag{17}$$

or

$$\boldsymbol{S}_{f,(12)3}=\frac{1}{A}\begin{pmatrix}\boldsymbol{S}_{f,1}+\boldsymbol{S}_{f,2}+\boldsymbol{S}_{f,3}-\frac{1}{2}\boldsymbol{S}_{f,1}\times\boldsymbol{S}_{f,2}-\frac{1}{2}\boldsymbol{S}_{f,1}\times\boldsymbol{S}_{f,3}-\frac{1}{2}\boldsymbol{S}_{f,2}\times\boldsymbol{S}_{f,3} \\ +\frac{1}{4}\left(\boldsymbol{S}_{f,1}\times\boldsymbol{S}_{f,2}\right)\times\boldsymbol{S}_{f,3}-(1-B)\boldsymbol{S}_{f,3}+\frac{1}{2}\boldsymbol{S}_{fp,12}\times\boldsymbol{S}_{f,3}-\boldsymbol{S}_{fp,12}-B\boldsymbol{S}_{fp,(12)3}\end{pmatrix} \tag{18}$$

where

$$A=1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2}-\tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,3}-\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\boldsymbol{s}_{f,2}^{\mathrm{T}}\boldsymbol{s}_{f,3}-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\boldsymbol{s}_{f,3}^{\mathrm{T}}\left(\boldsymbol{s}_{f,2}\times\boldsymbol{s}_{f,1}\right),$$

$$B=1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2},\quad \boldsymbol{S}_{fp,12}=\begin{pmatrix}\boldsymbol{0}\\ \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}t_2\boldsymbol{s}_{f,1}+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}t_1\boldsymbol{s}_{f,2}\end{pmatrix},$$

$$\boldsymbol{S}_{fp,(12)3}=\begin{pmatrix}\boldsymbol{0}\\ \tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2}t_3\boldsymbol{s}_{f,1}+\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}t_3\boldsymbol{s}_{f,2}+\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}t_3\left(\boldsymbol{s}_{f,2}\times\boldsymbol{s}_{f,1}\right) \\ +2\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\left(\boldsymbol{r}_{f,1}-\boldsymbol{r}_{f,2}\right)^{\mathrm{T}}\left(\boldsymbol{s}_{f,1}\times\boldsymbol{s}_{f,2}\right)\boldsymbol{s}_{f,3}+\left(\tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2}+\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2}\right)t_1\boldsymbol{s}_{f,3} \\ +\left(\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}+\tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2}\right)t_2\boldsymbol{s}_{f,3}\end{pmatrix}$$

Similarly, assuming that $\boldsymbol{S}_{f,23} = \boldsymbol{S}_{f,2} \triangle \boldsymbol{S}_{f,3}$ is the screw triangle product of the last two finite screws in Eq. (15), and replacing the subscripts ‘ (12)’ by ‘1’ and ‘3’ by ‘(23)’ in Eq. (17), yields

$$\boldsymbol{S}_{f,1(23)} = \boldsymbol{S}_{f,1} \triangle \boldsymbol{S}_{f,23} = \frac{1}{A}\begin{pmatrix} \boldsymbol{S}_{f,1} + \boldsymbol{S}_{f,2} + \boldsymbol{S}_{f,3} - \frac{1}{2}\boldsymbol{S}_{f,1} \times \boldsymbol{S}_{f,2} - \frac{1}{2}\boldsymbol{S}_{f,1} \times \boldsymbol{S}_{f,3} - \frac{1}{2}\boldsymbol{S}_{f,2} \times \boldsymbol{S}_{f,3} \\ +\frac{1}{4}\boldsymbol{S}_{f,1} \times \left(\boldsymbol{S}_{f,2} \times \boldsymbol{S}_{f,3}\right) - (1-C)\boldsymbol{S}_{f,1} + \frac{1}{2}\boldsymbol{S}_{f,1} \times \boldsymbol{S}_{fp,23} - \boldsymbol{S}_{fp,23} - C\boldsymbol{S}_{fp,1(23)} \end{pmatrix} \tag{19}$$

It is easy to prove that the following identity holds

$$\begin{aligned} &\frac{1}{4}\left(\boldsymbol{S}_{f,1} \times \boldsymbol{S}_{f,2}\right) \times \boldsymbol{S}_{f,3} - (1-B)\boldsymbol{S}_{f,3} + \frac{1}{2}\boldsymbol{S}_{fp,12} \times \boldsymbol{S}_{f,3} - \boldsymbol{S}_{fp,12} - B\boldsymbol{S}_{fp,(12)3} \\ &\equiv \frac{1}{4}\boldsymbol{S}_{f,1} \times \left(\boldsymbol{S}_{f,2} \times \boldsymbol{S}_{f,3}\right) - (1-C)\boldsymbol{S}_{f,1} + \frac{1}{2}\boldsymbol{S}_{f,1} \times \boldsymbol{S}_{fp,23} - \boldsymbol{S}_{fp,23} - C\boldsymbol{S}_{fp,1(23)} \end{aligned} \tag{20}$$

Thus

$$\boldsymbol{S}_{f,123} = \boldsymbol{S}_{f,(12)3} = \boldsymbol{S}_{f,1(23)} \tag{21}$$

This means that the screw triangle product satisfies the associativity, i.e. in the composition of a set of successive finite screws, they can be divided into a number of groups first so long as their sequences within and amongst groups are unchanged. In this way, we can prove that the entire set of finite screws under the screw triangle product is of a closed and associative algebraic structure.

3.2 The derivative law of screw triangle product

In order to investigate the differential property of screw triangle product, let the screw triangle product of two finite screws be written as

$$\boldsymbol{S}_{f,12} = \boldsymbol{S}_{f,1} \triangle \boldsymbol{S}_{f,2} = \frac{\boldsymbol{E}}{F} \tag{22}$$

where

$$\boldsymbol{E} = \left(\boldsymbol{S}_{f,1} + \boldsymbol{S}_{f,2} - \frac{1}{2}\boldsymbol{S}_{f,1} \times \boldsymbol{S}_{f,2} - G\left(t_2\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_{f,1}\end{pmatrix} + t_1\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_{f,2}\end{pmatrix}\right)\right), \quad F = 1 - G\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2}, \quad G = \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}$$

Taking time derivative of Eq. (22) and noting that $\boldsymbol{s}_{f,j}$ and $\boldsymbol{r}_{f,j}$ ($j = 1, 2$) are constant vectors, leads to

$$\dot{\boldsymbol{S}}_{f,12} = \frac{\dot{\boldsymbol{E}}}{F} - \frac{\boldsymbol{E}\dot{F}}{F^2} \tag{23}$$

where

$$\dot{\boldsymbol{E}} = \left(\dot{\boldsymbol{S}}_{f,1} + \dot{\boldsymbol{S}}_{f,2} - \frac{1}{2}\left(\dot{\boldsymbol{S}}_{f,1} \times \boldsymbol{S}_{f,2} + \boldsymbol{S}_{f,1} \times \dot{\boldsymbol{S}}_{f,2}\right) - \dot{G}\left(t_2\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_{f,1}\end{pmatrix} + t_1\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_{f,2}\end{pmatrix}\right) + G\left(\dot{t}_2\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_{f,1}\end{pmatrix} + \dot{t}_1\begin{pmatrix}\boldsymbol{0} \\ \boldsymbol{s}_{f,2}\end{pmatrix}\right)\right), \quad \dot{F} = -\dot{G}\boldsymbol{s}_{f,1}^{\mathrm{T}}\boldsymbol{s}_{f,2}$$

At the initial pose where $\theta_j = 0$ and $t_j = 0$, we have

$$\boldsymbol{E}=\boldsymbol{0},\quad G=0,\quad F=1,\quad \dot{G}=0,\quad \dot{F}=0,\quad \dot{\boldsymbol{S}}_{f,1}\times\boldsymbol{S}_{f,2}+\boldsymbol{S}_{f,1}\times\dot{\boldsymbol{S}}_{f,2}=\boldsymbol{0}$$

Thus

$$\dot{\boldsymbol{S}}_{f,12}\Big|_{\substack{\theta_j=0\\ t_j=0}, j=1,2}=\dot{\boldsymbol{S}}_{f,1}\Big|_{\substack{\theta_j=0\\ t_j=0}, j=1,2}+\dot{\boldsymbol{S}}_{f,2}\Big|_{\substack{\theta_j=0\\ t_j=0}, j=1,2}=\boldsymbol{S}_{t,1}+\boldsymbol{S}_{t,2} \tag{24}$$

Note that $\dot{\boldsymbol{S}}_{f,12}$ can also be rewritten into the standard form given in Eq. (6), it thereby satisfies the closure of instantaneous motion composition. Hence, the property given in Eq. (24) can be extended to a general form for the system composed of n 1-DOF motion generators

$$\dot{\boldsymbol{S}}_{f,12\ldots n}\Big|_{\substack{\theta_j=0\\ t_j=0}, j=1,2,\cdots,n}=\sum_{j=1}^{n}\dot{\boldsymbol{S}}_{f,j}\Big|_{\substack{\theta_j=0\\ t_j=0}, j=1,2,\cdots,n}=\sum_{j=1}^{n}\boldsymbol{S}_{t,j} \tag{25}$$

where

$$\boldsymbol{S}_{t,j}=\dot{\theta}_j\begin{pmatrix}\boldsymbol{s}_{t,j}\\ \boldsymbol{r}_{t,j}\times\boldsymbol{s}_{t,j}\end{pmatrix}+\dot{t}_j\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_{t,j}\end{pmatrix}=\begin{pmatrix}\boldsymbol{\omega}_j\\ \boldsymbol{v}_j\end{pmatrix},\quad j=1,2,\cdots,n$$

It is clear that any curve in the entire set of finite screws is continuous and differentiable, and its time derivative forms an instantaneous screw system. The foregoing associativity and derivative properties allow the algebraic structures of finite and instantaneous screws as addressed in what follows.

4. Algebraic Structures of Finite and Instantaneous Screws

On the basis of Sections 2 and 3, we intend to reveal the specific algebra structures of finite and instantaneous screws. Let $\{\boldsymbol{S}_f\}$ be the entire set of finite screws in the form given by Eq. (3)

$$\{\boldsymbol{S}_f\}=\left\{2\tan\frac{\theta}{2}\begin{pmatrix}\boldsymbol{s}_f\\ \boldsymbol{r}_f\times\boldsymbol{s}_f\end{pmatrix}+t\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{s}_f\end{pmatrix}\middle|\, \boldsymbol{s}_f,\boldsymbol{r}_f\in\mathbb{R}^3,\ |\boldsymbol{s}_f|=1,\ \theta,t\in\mathbb{R}\right\} \tag{26}$$

In the above sections, the closure of this set under screw triangle product and the associativity and derivative properties of this product are proved. And let $\boldsymbol{S}_{f0}\in\{\boldsymbol{S}_f\}$ at $\theta=0$ and $t=0$ be the identity element of $\{\boldsymbol{S}_f\}$, i.e.

$$\boldsymbol{S}_{f,0}=\begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{0}\end{pmatrix} \tag{27}$$

Note that $-\boldsymbol{S}_f\in\{\boldsymbol{S}_f\}$ for $\forall\boldsymbol{S}_f\in\{\boldsymbol{S}_f\}$ such that

$$\boldsymbol{S}_f\triangle\boldsymbol{S}_{f,0}=\boldsymbol{S}_{f,0}\triangle\boldsymbol{S}_f=\boldsymbol{S}_f,$$

$$\boldsymbol{S}_f\triangle(-\boldsymbol{S}_f)=(-\boldsymbol{S}_f)\triangle\boldsymbol{S}_f=\boldsymbol{S}_{f,0} \tag{28}$$

This means that there exists a unique inverse for each finite screw.

Building upon the above proofs and analyses, we can conclude that the algebraic structure of $\{\boldsymbol{S}_f\}$ is a Lie group under the screw triangle product according to the Lie group theory [44-45]. Here, we refer $\{\boldsymbol{S}_f\}$ to as Lie group of finite screws. Obviously, a set of one-parameter finite screws given by Eq. (7) forms a subgroup of this Lie group.

In the neighborhood of the identity, let $\{\boldsymbol{S}_t\} = \left\{ \dot{\boldsymbol{S}}_f \big|_{\substack{\theta=0\\t=0}} \right\}$ be the entire set of instantaneous screws given in Eq. (6)

$$\{\boldsymbol{S}_t\} = \left\{ \dot{\boldsymbol{S}}_f \big|_{\substack{\theta=0\\t=0}} \right\} = \left\{ \dot{\theta}\begin{pmatrix} \boldsymbol{s}_t \\ \boldsymbol{r}_t \times \boldsymbol{s}_t \end{pmatrix} + \dot{t}\begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{s}_t \end{pmatrix} \middle| \boldsymbol{s}_t, \boldsymbol{r}_t \in \mathbb{R}^3,\ |\boldsymbol{s}_t| = 1,\ \dot{\theta}, \dot{t} \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \boldsymbol{\omega} \\ \boldsymbol{v} \end{pmatrix} \middle| \boldsymbol{\omega}, \boldsymbol{v} \in \mathbb{R}^3 \right\} \tag{29}$$

This set spans a vector space having the following properties under screw cross product: bilinear (Eqs. (30) and (31)), noncommutative (Eq. (32)) and Jocobi identity (Eq. (33))

$$\left(\boldsymbol{S}_{t,1} + \boldsymbol{S}_{t,2}\right) \times \boldsymbol{S}_{t,3} = \boldsymbol{S}_{t,1} \times \boldsymbol{S}_{t,3} + \boldsymbol{S}_{t,2} \times \boldsymbol{S}_{t,3},$$

$$\boldsymbol{S}_{t,1} \times \left(\boldsymbol{S}_{t,2} + \boldsymbol{S}_{t,3}\right) = \boldsymbol{S}_{t,1} \times \boldsymbol{S}_{t,2} + \boldsymbol{S}_{t,1} \times \boldsymbol{S}_{t,3} \tag{30}$$

$$\left(k\boldsymbol{S}_{t,1}\right) \times \boldsymbol{S}_{t,2} = \boldsymbol{S}_{t,1} \times \left(k\boldsymbol{S}_{t,2}\right) = k\left(\boldsymbol{S}_{t,1} \times \boldsymbol{S}_{t,2}\right), \quad \forall k \in \mathbb{R} \tag{31}$$

$$\boldsymbol{S}_{t,1} \times \boldsymbol{S}_{t,2} = -\boldsymbol{S}_{t,2} \times \boldsymbol{S}_{t,1} \tag{32}$$

$$\boldsymbol{S}_{t,1} \times \left(\boldsymbol{S}_{t,2} \times \boldsymbol{S}_{t,3}\right) + \boldsymbol{S}_{t,2} \times \left(\boldsymbol{S}_{t,3} \times \boldsymbol{S}_{t,1}\right) + \boldsymbol{S}_{t,3} \times \left(\boldsymbol{S}_{t,1} \times \boldsymbol{S}_{t,2}\right) = \boldsymbol{0} \tag{33}$$

Thus, $\{\boldsymbol{S}_t\}$ is a Lie algebra under the screw cross product in the light of [44-45]. So, we refer $\{\boldsymbol{S}_t\}$ to as Lie algebra of instantaneous screws.

Finally, consider a continuous curve $\{\boldsymbol{S}_f\}^C \subseteq \{\boldsymbol{S}_f\}$ composed by n Lie subgroups of one-parameter finite screws, i.e.

$$\{\boldsymbol{S}_f\}^C = \left\{\boldsymbol{S}_{f,12\cdots n}\right\} = \left\{\boldsymbol{S}_{f,1} \triangle \boldsymbol{S}_{f,2} \triangle \cdots \triangle \boldsymbol{S}_{f,n}\right\} \tag{34}$$

Note that the time derivative of $\boldsymbol{S}_{f,12\cdots n} \in \{\boldsymbol{S}_f\}^C$ at the neighborhood of the identity element has the form (see Eq. (25))

$$\dot{\boldsymbol{S}}_{f,12\ldots n}\Big|_{\substack{\theta_j=0\\t_j=0}, j=1,2,\cdots,n} = \sum_{j=1}^{n} \boldsymbol{S}_{t,j} \tag{35}$$

Thus, the tangent space of $\{\boldsymbol{S}_f\}^C$ in the neighborhood of the identity element forms an instantaneous screw sub-system of $\{\boldsymbol{S}_t\}$. So, we conclude that $\{\boldsymbol{S}_t\}$ is the corresponding Lie algebra of Lie group $\{\boldsymbol{S}_f\}$.

5. Conclusions

This paper presents a way of relating instantaneous and finite screws based on screw triangle product. The following conclusions are drawn.

1) The screw triangle product of a number of one-parameter finite screws in 6×1 quasi-vector form satisfies the associativity, and its time derivative at initial pose can be expressed as a linear combination of instantaneous screws in 6×1 vector form.

2) By examining the closure, associativity, existence of an identity, and existence of inverses of the entire set of finite

screws under screw triangle product, we have proven that the algebraic structure of the set forms a Lie group, and its time derivative at the identity forms the corresponding Lie algebra of instantaneous screws under the screw cross product.
3) The revealed relationship between finite and instantaneous screws provides a possibility to model finite and instantaneous motions in a consistent manner, allowing type synthesis and performance analysis of robotic mechanisms to be unified into the concise framework of screw theory.

Acknowledgements

This research work is partially supported by the National Natural Science Foundation of China (NSFC) under Grant 51420105007 and 51535008, and Tianjin Research Program of Application Foundation and Advanced Technology under Grant 16JCYBJC19300.

References

[1] R. S. Ball, A Treatise on the Theory of Screws, Cambridge University Press, 1900.
[2] K. H. Hunt, Kinematic Geometry of Mechanisms, Oxford University Press, 1978.
[3] J. Duffy, The Screw Theory and Its Application, Class Note of University of Florida, 1982.
[4] J. Angeles, Fundamentals of Robotic Mechanical Systems: Theory, Methods, and Algorithms, Mechanical Engineering Series, 3rd ed., Vol. 1, Springer, 2007.
[5] S. A. Joshi, L. W. Tsai, Jacobian analysis of limited-DOF parallel manipulators, ASME J. Mech. Des. 124 (2) (2002) 254-258.
[6] T. Huang, H. T. Liu, D. G. Chetwynd, Generalized Jacobian analysis of lower mobility manipulators, Mech. Mach. Theory 46 (6) (2011) 831-844.
[7] J. S. Dai, J. Rees, Interrelationship between screw systems and corresponding reciprocal systems and applications, Mech. Mach. Theory 36 (5) (2001) 633-651.
[8] J. S. Dai, J. Rees, Null space construction using cofactors from a screw algebra context, Proc. Royal Society London A: Mathematical, Physical and Engineering Sciences 458 (2024) (2002) 1845-1866.
[9] J. S. Dai, Z. Huang, H. Lipkin, Mobility of overconstrained parallel mechanisms, ASME J. Mech. Des. 128 (1) (2004) 220-229.
[10] Z. Huang, Q. C. Li, General methodology for type synthesis of symmetrical lower-mobility parallel manipulators and several novel manipulators, Int. J. Robot. Res. 21 (2) (2002) 131-145.
[11] Y. F. Fang, L. W. Tsai, Structure synthesis of a class of 4-DoF and 5-DoF parallel manipulators with identical limb structures, Int. J. Robot. Res. 21 (9) (2002) 799-810.
[12] X. W. Kong, C. M. Gosselin, Type Synthesis of Parallel Mechanisms, Springer, 2007.
[13] J. Meng, G. F. Liu, Z. X. Li, A geometric theory for analysis and synthesis of sub-6 DoF parallel manipulators, IEEE Trans. Robot. 23 (4) (2007) 625-649.
[14] T. L. Yang, A. X. Liu, Q. Jin, Y. F. Luo, H. P. Shen, L. B. Hang, Position and orientation characteristic equation for topological design of robot mechanisms, ASME J. Mech. Des. 131 (2) (2008) 021001-1-021001-17.
[15] S. F. Yang, T. Sun, T. Huang, Q. C. Li, D. B. Gu, A finite screw approach to type synthesis of three-DOF translational parallel mechanisms, Mech. Mach. Theory 104 (2016) 405-419.
[16] F. M. Dimentberg, The Screw Calculus and Its Application to Mechanics, Moscow, 1965, English Translation, U.S. Department of Commerce, (N.T.I.S), No. AD, 1969, pp. 680–993.
[17] J. S. Dai, Historical relation between mechanisms and screw theory and the development of finite displacement screws, Journal of Mechanical Engineering 51 (13) (2015) 13-26.
[18] B. Roth, On the screw axes and other special lines associated with spatial displacements of a rigid body, ASME Journal of Manufacturing Science and Engineering 89 (1) (1967) 102-110.
[19] I. A. Parkin, A third conformation with the screw systems: finite twist displacements of a directed line and point, Mech. Mach. Theory (27) (2) (1992) 177-188.

[20] C. T. Huang, The finite screw systems associated with a prismatic-revolute dyad and the screw displacement of a point, Mech. Mach. Theory 29 (8) (1994) 1131-1142.

[21] C. T. Huang, C. M. Chen, The linear representation of the screw triangle—a unification of finite and infinitesimal kinematics, ASME J. Mech. Des. 117 (4) (1995) 554-560.

[22] J. S. Dai, N. Holland, D. R. Kerr, Finite twist mapping and its application to planar serial manipulators with revolute joints, Proc. IMechE Part C: J. Mech. Eng. Sci. 209 (4) (1995) 263-271.

[23] J. S. Dai, An historical review of the theoretical development of rigid body displacements from Rodrigues parameters to the finite twist, Mech. Mach. Theory 41 (1) (2006) 41-52.

[24] D. Zarrouk, M. Shoham, A note on the screw triangle, ASME J. Mech. Robot. 3 (1) (2011) 014502-1-014502-4.

[25] J. S. Dai, Finite displacement screw operators with embedded Chasles' motion, ASME J. Mech. Robot. 4 (4) (2012) 041002-1-041002-9.

[26] J. S. Dai, Geometrical Foundations and Screw Algebra for Mechanisms and Robotics, Higher Education Press, Beijing, 2014, ISBN: 9787040334838 (translated from J. S. Dai, Screw Algebra and Kinematic Approaches for Mechanisms and Robotics, to be published by Springer, London, 2016).

[27] C. T. Huang, The cylindroid associated with finite motions of the Bennett mechanism, ASME J. Mech. Des. 119 (4) (1997) 521-524.

[28] K. H. Hunt, I. A. Parkin, Finite displacements of points, planes, and lines via screw theory, Mech. Mach. Theory 30 (2) (1995) 177-192.

[29] J. P. Merlet, C. M. Gosselin, T. Huang, Handbook of Robotics, Chapter 18: Parallel Mechanisms, Springer, 2014.

[30] F. Klein, A comparative review of recent researches in geometry, Mathematische Annalen 43 (1893) 63-100.

[31] R. W. Brockett, Robotic manipulators and the product of exponential formula, in Mathematical Theory of Networks and Systems, Springer, 1984, 120-129.

[32] R. Murray, Z. X. Li, S. Sastry, A Mathematical Introduction to Robotic Manipulation, CRC Press, 1994.

[33] P. Fanghella, C. Galletti, Mobility analysis of single-loop kinematic chains: an algorithmic approach based on displacement groups, Mech. Mach. Theory 29 (8) (1994) 1187-1204.

[34] C. C. Lee, J. M. Hervé, Parallel mechanisms generating 3-DoF finite translation and (2 or 1)-DoF infinitesimal rotation, Mech. Mach. Theory 51 (5) (2012) 185-194.

[35] J. S. Dai, Euler-Rodrigues formula variations, quaternion conjugation and intrinsic connections, Mech. Mach. Theory 92 (2015) 144-152.

[36] W. R. Hamilton, Elements of Quaternions, Cambridge University Press, 1899.

[37] A. Perez, J. M. McCarthy, Dual quaternion synthesis of constrained robotic systems, ASME J. Mech. Des. 126 (3) (2004) 425-435.

[38] J. M. Selig, Geometric Fundamentals of Robotics, 2nd ed, Springer, 2005.

[39] J. M. Selig, Exponential and cayley maps for dual quaternions, Advances in Applied Clifford Algebras 20 (3) (2010) 923-936.

[40] J. M. Selig, E. Bayro, Rigid body dynamics using Clifford algebra, Advances in Applied Clifford Algebras 20 (1) (2010) 141-154.

[41] R. Featherstone, Spatial Vector Algebra, Chapter: Rigid Body Dynamics Algorithms, Springer, 2008.

[42] O. Bottema, B. Roth, Theoretical Kinematics, Dove, 1990.

[43] C. T. Huang, K. Sugimoto, I. A. Parkin, The correspondence between finite screw systems and projective spaces, Mech. Mach. Theory 43 (1) (2008) 50-56.

[44] C. Chevalley, Theory of Lie groups, Princeton University Press, 1946.

[45] B. C. Hall, Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, Springer, 2003.